Globally hyperbolic spacetimes can be defined as "causal" instead of "strongly causal"

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Abstract

The classical definition of *global hyperbolicity* for a spacetime (M, g) comprises two conditions: (A) compactness of the diamonds $J^+(p) \cap J^-(q)$, and (B) strong causality. Here we show that condition (B) can be replaced just by causality. In fact, we show first that the classical definition of causal simplicity (which impose to be distinguishing, apart from the closedness of $J^+(p)$, $J^-(q)$) can be weakened in causal instead of distinguishing. So, the full consistency of the causal ladder (recently proved by the authors in a definitive way) yields directly the result.

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1 Introduction

Global hyperbolicity is the most important condition on Causality, which lies at the top of the so-called causal hierarchy of spacetimes and is involved in problems as Cosmic Censorship, predictability etc. There are different alternative definitions of what global hyperbolicity means, but perhaps the most standard one is the following. A spacetime (M, g) is said *globally hyperbolic* if and only if it satisfies two conditions: (A) compactness of $J^+(p) \cap J^-(q)$ for all $p, q \in M$ (i.e. no "naked" singularity can exist) and (B) strong causality (no "almost closed" causal curve exists).

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In this note, we stress the possibility to simplify the definition of global hyperbolicity, by weakening the requirement of *strong causality* into *causality* (i.e. no closed causal curve exist), which is both, mathematically simpler and physically clearer. More in depth, we will see:

1. The usual definition of causal simplicity for a spacetime (M, g) can be simplified, just by imposing:

   (a) closedness of $J^{\pm}(p)$ for all p, and

   (b) causality.

2. In any spacetime, the condition:

   (A) compactness of $J^{+}(p) \cap J^{-}(q)$ for all $p, q \in M$

   always implies previous condition (a).

3. Thus, any spacetime which satisfies (A) and (b) is causally simple. As this condition implies strong causality, *any spacetime which satisfies (A) and (b) is strongly causal and, then, globally hyperbolic*.

About this last point, it is worth pointing out that there were some problems on the full consistency of the causal hierarchy of spacetime, which has been recently solved ([2, 3], see also the reviews [9, 6]). Concretely (see Remark 3.3 below), the so-called "folk questions" on smoothability posed a question on the equivalence between two alternative definitions of *stable causality*, i.e., the level in the standard hierarchy of causality which is immediately more restrictive than strong causality. Nevertheless, after the solution of these folk problems, no doubt can exist in the use of the full consistency and its classical implications (a detailed study is done in the survey [6]).

In the next Sections 2, 3 we prove and discuss these simplifications of causal simplicity and global hyperbolicity, respectively. All the notation and concepts are standard, and can be found, for example, in [1, 5, 7, 8, 10, 11].

2 The optimal definitions of causal simplicity

Let us start with the following result:

Proposition 2.1. *Assume that (M, g) satisfies:*

(a) $J^{+}(p)$ and $J^{-}(p)$ are closed for all $p, q \in M$.

Then the following two conditions are equivalent:

(b1) (M, g) is causal,

(b2) (M, g) is distinguishing, i.e., if $p \neq q$ then $I^{+}(p) \neq I^{+}(q)$ and $I^{-}(p) \neq I^{-}(q)$.

Proof. Let us prove that, under (a), hypothesis $(b1)$ imply $(b2)$ (the converse holds always and is well-known). Otherwise, if $p \neq q$ and, say $I^{+}(p) = I^{+}(q)$, choose any sequence $\{q_n\} \to q$, with $q \ll q_n$. Then, $q \in J^{+}(p)$ because $q \in \bar{I}^{+}(q) = \bar{I}^{+}(p) = \bar{J}^{+}(p) = J^{+}(p)$ (the first equality holds because the distinguishing property fails for p and q, the second one holds always, and the last one by hypothesis (a)). Analogously, $p \in J^+(q)$, i.e., $p < q < p$ and the spacetime is not causal, a contradiction. $\square$

Remark 2.1. The classical definition of causal simplicity [1, p. 65], [10, Dfn. 2.17] assumes[1] (a) and $(b2)$. But $(b2)$ is always more restrictive than $(b1)$ and, thus, it is natural to choose a definition of causal simplicity with minimum hypotheses, that is, *a causally simple spacetime is the one which satisfies (a) and (b1).* Even more, it is easy to show that the requirement $(b1)$ cannot be weakened in (M, g) chronological (see the cylinder C constructed in item 2 below Remark 3.3).

3 The optimal definition of global hyperbolicity

The following result is rather standard in Causality, in order to prove that a globally hyperbolic spacetime is causally simple (see for example [10, Prop. 2.26]):

Lemma 3.1. *Assume that a spacetime (M, g) satisfies:*

(A) $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$.

Then it also satisfies:

(a) $J^+(p)$ and $J^-(p)$ are closed for all $p, q \in M$.

Proof. Assume that $J^+(p)$ is not closed and choose $r \in \bar{J}^+(p) \backslash J^+(p)$ and $q \in I^+(r)$. Take a sequence $\{r_n\} \to r$ with $r_n \in I^+(p)$ for all n, and notice that $r_n \ll q$ up to a finite number of n (as $I^-(q) \ni r$ is open). Thus, $\{r_n\}_n \subset J^+(p) \cap J^-(q)$, which is compact, but converges to the point r, which does not lie in this subset, a contradiction. $\square$

Now, the next result yields the simplification of global hyperbolicity; it becomes a straightforward consequence of previous ones and the full consistency of the causal ladder of causality.

Theorem 3.2. *Assume that (M, g) satisfies:*

(A) $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$.

Then the following two conditions are equivalent:

(B1) (M, g) is causal, i.e., there are no closed causal curves.

(B2) (M, g) is strongly causal, i.e., for any $p \in M$, given any neighborhood U of p there exists a neighborhood $V \subset U$, $p \in V$, such that any future-directed (and hence also any past-directed) causal curve $\gamma : [a, b] \to M$ with endpoints at V is entirely contained in U.

[1] In [5, p. 188], there is a small gap, because no condition type (b) is assumed (in this case, any totally vicious spacetime would be causally simple).

Proof. By Lemma 3.1 (A) plus (B1) imply conditions (a) plus $(b1)$ in Proposition 2.1 i.e., causal simplicity. This is a more restrictive level of the causal ladder than strong causality (a standard reference is [1, p. 73], but see discussion below), and thus, the result follows. □

Remark 3.3. Notice that, in Theorem 3.2, the proof of strong causality follows indirectly, as a consequence of causal simplicity. Taking into account the known properties of the hierarchy of spacetimes, if (M, g) is causally simple then it is *causally continuous* and, thus, the volume functions $t^{\pm}(p) = m(J^{\pm}(p))$ (defined for any *admissible measure* m), are -continuous- time functions. In particular, (M, g) satisfies one of the two definitions[2] of *stable causality*, and it is strongly causal. The full consistency between these two alternative definitions (and, thus, of the whole causal ladder) has been obtained recently [3, 2]; we refer to [9, 6] for details and subtleties.

Now, we emphasize the following items:

1. The classical definition of global hyperbolicity ([1, p. 65], [5, p. 206], [7, p. 412], [8, p. 48], [10, Dfn. 2.17], [11, p. 209]) impose (A) and (B2). Nevertheless, (B2)⇒ (B1) always trivially and, thus, it is natural to choose a definition of global hyperbolicity with minimum hypotheses, that is, *a globally hyperbolic spacetime is the one which satisfies (A) and (B1).*

2. The requirement (B1), i.e., the spacetime is causal, cannot be weakened in (M, g) chronological. In fact, the cylinder C obtained as the quotient from Lorentz-Minkowski spacetime in null coordinates $(\mathbb{R}^2, g = 2dudv)$ by the group generated by the translation $(u, v) \to (u+1, v)$ is chronological, satisfies (A) but it is not causal.

3. An alternative hypothesis to condition (A) is:

   (A') For each $p, q \in M$, the space of future-directed causal curves (continuous, and up to a strictly increasing reparameterization) which connect p with q, $C(p, q)$, is compact in the C^0 topology.

   So, conditions (A') and (B1) are also equivalent to global hyperbolicity. Even more, the use of the C^0 topology makes sense only when causality holds (otherwise, if γ is a closed causal curve through p, giving more and more rounds one would obtain non-equivalent curves; clearly, it would be natural to consider a topology such that $C(p, p)$ is not compact). So, essentially global hyperbolicity reduces to (A').

4. Due to a classical theorem by Geroch [4], globally hyperbolic spacetimes are alternatively defined as the ones which admit a Cauchy hypersurface. Even more, the recent solution of the "folk" questions on smoothability [3], yield further orthogonal metric structures for globally hyperbolic spacetimes. Nevertheless, these alternative possibilities are formulated under hypotheses different to (A) and (B1), (B2) and, in principle, they are not affected by Theorem 3.2 (even though some relations are still possible;

[2] The other essentially different definition is a spacetime which remains causal after opening a bit all the causal cones.

for example, recall that if S is an achronal subset then $D(S)$ is strongly causal).

References

[1] Beem, J. K., Ehrlich, P. E., and Easley, K. L.: *Global Lorentzian Geometry.* New York: Marcel Dekker Inc. (1996)

[2] Bernal, A. N. and Sánchez, M.: On smooth Cauchy hypersurfaces and Geroch's splitting theorem. Commun. Math. Phys. **243**, 461–470 (2003)

[3] Bernal, A. N. and Sánchez, M.: Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes. *Commun. Math. Phys.* **257**, 43–50 (2005), gr-qc/0401112.

[4] Geroch, R.: Domain of dependence, *J. Math. Phys.* **11**, 437–449 (1970)

[5] Hawking, S. W. and Ellis, G. F. R.: *The Large Scale Structure of Space-Time.* Cambridge: Cambridge University Press (1973)

[6] Minguzzi, E. and Sánchez, M.: *The causal hierarchy of spacetimes*, European Mathematical Society Publishing House, vol. 'Recent developments in pseudo-Riemannian Geometry' based on the semester 'Geometry of pseudo-Riemannian manifolds with application to physics' at ESI, Vienna, Sept - Dec 2005, submitted, gr-qc/0609119

[7] O'Neill, B.: *Semi-Riemannian Geometry.* San Diego: Academic Press (1983)

[8] Penrose, R.: *Techniques of Differential Topology in Relativity.* Cbms-Nsf Regional Conference Series in Applied Mathematics. Philadelphia: SIAM (1972)

[9] Sánchez, M.: Causal hierarchy of spacetimes, temporal functions and smoothness of Geroch's splitting: a revision. *Contemporanea Matematica* **28**, 127–155 (2005), gr-qc/0411143.

[10] Senovilla, J. M. M.: Singularity theorems and their consequences. Gen. Relativ. Gravit. **29**, 701–848 (1997)

[11] Wald, R. M.: *General Relativity.* Chicago: The University of Chicago Press (1984)